Nested Grassmanns for Dimensionality Reduction with Applications to Shape Analysis

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Main Contributions

- We propose a general framework for constructing a nested sequence of Riemannian homogeneous spaces and study the nested Grassmanns specifically.
- We apply the nested Grassmanns to the dimensionality problem in planar shape analysis.

Netsed Homogeneous Spaces

Homogeneous Spaces

- A Riemannian manifold (M, g) is called a homogeneous space if there exists a group G together with a *transitive* group action $a : G \times M \to M$ (i.e., for any $x, y \in M$, there exists $h \in G$ such that $a(h, x) = y$) and the Riemannian metric g is invariant to this group action.
- The manifold M can be expressed as $M \cong G/H$ where $H = \{h \in G : a(h, p) = p\}$ is the isotropy subgroup of G at $p \in G$.
- Examples:
  1. Spheres: $S^{n-1} \cong \mathrm{SO}(n)/\mathrm{SO}(n-1)$,
  2. Grassmann manifolds: $\mathrm{Gr}(p, \mathbb{R}^n) \cong \mathrm{O}(n)/(\mathrm{O}(p) \times \mathrm{O}(n-p))$, etc.

Recipe for constructing nested homogeneous spaces:

1. Define an embedding $\tilde{\iota} : G_m \to G_{m+1}$.
2. Find the submersion $\psi : G_m \to G_m/H_m$ and the identification map $f : G_m/H_m \to M_m$.
3. Induce the embedding $\iota : M_m \to M_{m+1}$ from $\tilde{\iota}$, ψ, and f.

$$\begin{tikzcd}
G_m \arrow[r, "\tilde{\iota}"] \arrow[d, "\psi"] & G_{m+1} \arrow[d, "\psi"] \\
G_m/H_m & G_{m+1}/H_{m+1} \\
M_m \arrow[u, "f"] \arrow[r, "\iota"] & M_{m+1} \arrow[u, "f"]
\end{tikzcd}$$

Figure 1: Commutative diagram of the induced embedding for homogeneous spaces.

Nested Grassmanns

- The Grassmann manifold $\mathrm{Gr}(p, \mathbb{V})$, where $\mathbb{V}$ is a vector space and $1 \le p \le \dim(\mathbb{V})$, is the manifold of all p-dimensional subspaces of $\mathbb{V}$. Usually, $\mathbb{V} = \mathbb{R}^n$ or $\mathbb{V} = \mathbb{C}^n$. We write $\mathrm{Gr}(p, m)\mathrm{Gr}(p, \mathbb{R}^m)$.
- For $\mathcal{X} \in \mathrm{Gr}(p, \mathbb{V})$, we can represent $\mathcal{X}$ by its orthonormal basis, i.e., $\mathcal{X} = \mathrm{span}(X)$.
- For $\mathcal{X}, \mathcal{Y} \in \mathrm{Gr}(p, \mathbb{V})$, the *geodesic* distance between $\mathcal{X}$ and $\mathcal{Y}$ is $d_g(\mathcal{X}, \mathcal{Y}) = \sqrt{\sum_{i=1}^p \theta_i^2}$ where θ_i's are the principal angles between $\mathcal{X}$ and $\mathcal{Y}$.
- The projection distance (or the chordal distance) is given by $d_p(\mathcal{X}, \mathcal{Y}) = \sqrt{\sum_{i=1}^p \sin^2 \theta_i}$.

Construction

- $\mathrm{Gr}(p, m) \cong \mathrm{O}(m)/(\mathrm{O}(p) \times \mathrm{O}(m-p))$.
- The induced embedding $\iota_m : \mathrm{Gr}(p, m) \to \mathrm{Gr}(p, m+1)$ is $\iota_m(\mathcal{X}) = \mathrm{span}(\tilde{R}X + vb^T)$ where $b \in \mathbb{R}^p$, $\tilde{R} \in \mathrm{St}(m, m+1)$, and $v \in S^m$ is such that $\tilde{R}^T v = 0$.

Properties

1. If $b = 0$, then ι_m is an isometric embedding.
2. The projection $\pi_m : \mathrm{Gr}(p, m+1) \to \mathrm{Gr}(p, m)$ corresponding to ι_m is given by $\pi_m(\mathcal{X}) = \mathrm{span}(\tilde{R}^T X)$.
3. The embedding of $\mathrm{Gr}(p, m)$ into $\mathrm{Gr}(p, n)$ for $m < n$ is given by $\iota_{A,B}(\mathcal{X}) = \mathrm{span}(AX + B)$ where $A \in \mathrm{St}(m, n)$ and $B \in \mathbb{R}^{n \times p}$ such that $A^T B = 0$.
4. The corresponding projection from $\mathrm{Gr}(p, n)$ to $\mathrm{Gr}(p, m)$ is given by $\pi_A = \mathrm{span}(A^T X)$.

Dimensionality Reduction with Nested Grassmanns

Unsupervised

Given $\mathcal{X}_1, \ldots, \mathcal{X}_N \in \mathrm{Gr}(p, n)$, the desired projection map π is obtained by the minimizing the reconstruction error, i.e.,

$$A, B = \operatorname*{arg\,min}_{A \in \mathrm{St}(m,n), B \in \mathbb{R}^{n \times p}, A^T B = 0} \frac{1}{N} \sum_{i=1}^N d^2(\mathcal{X}_i, \widehat{\mathcal{X}}_i), \quad \widehat{\mathcal{X}}_i = \iota_{A,B}(\pi_A(\mathcal{X}_i))$$

where d is a distance metric on $\mathrm{Gr}(p, n)$.

Supervised

For labelled data $(\mathcal{X}_i, y_i)$ where $\mathcal{X}_i \in \mathrm{Gr}(p, n)$ and $y_i \in \{1, \ldots, k\}$,

$$A = \operatorname*{arg\,min}_{A \in \mathrm{St}(m,n)} \frac{1}{N^2} \sum_{i,j=1}^N a(\mathcal{X}_i, \mathcal{X}_j) d^2(\pi_A(\mathcal{X}_i), \pi_A(\mathcal{X}_j))$$

where $a(\cdot, \cdot)$ is the *affinity function* suggested by Harandi et al. (2018, Sec 3.1, Eq. (14)-(16)).

Simulation Studies

Comparison among Distance Metrics

- We generate $N = 50$ points on $\mathrm{Gr}(1, 10)$ and project them to $\mathrm{Gr}(1, 3)$. Then we compare the performance of the two distance metrics (geodesic and projection) under varying levels of variance σ, ranging from 1 to 10.

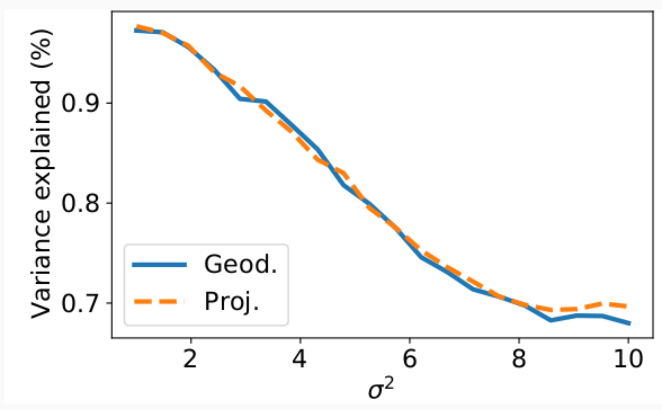

Figure 2: Comparison of the NG representations based on the projection and geodesic distances using the ratio of expressed variance[1].

Comparison of Nested Grassmanns (NG) and Principal Geodesic Analysis (PGA)

We compare NG and PGA under two scenarios:

- projecting $N = 50$ points on $\mathrm{Gr}(2, 30)$ to an $\tilde{m}$-dimensional submanifold (see the table below) and
- projecting $N = 50$ points on $\mathrm{Gr}(2, 10)$ to a 2-dimensional submanifold under varying variance σ (see the figure below).

	$\tilde{m}$ (dim. of submanifold)				
	2	4	6	8	10
NG	33.12%	50.49%	59.98%	67.85%	73.77%
PGA	16.36%	29.41%	40.81%	50.63%	59.29%

Table 1: The percentage of explained variance by PGA and NG representations respectively.

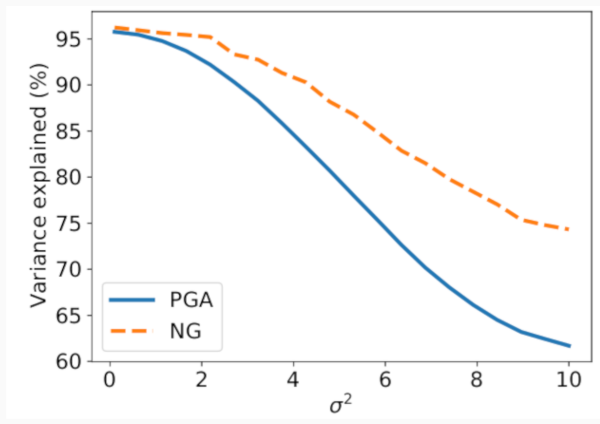

Figure 3: Comparison of NG and PGA algorithms via percentage of explained variance.

OASIS Datasets Experiments – Setup

- The OASIS database (Marcus et al., 2007) is a publicly available database that contains T1-MR brain scans of subjects of age ranging from 18 to 96.
- We randomly choose 4 brain scans within each decade, totalling 36 brain scans.
- From each scan, the Corpus Callosum (CC) region is segmented and 250 points are taken on the boundary of the CC region.
- In this case, the shape space is $\Sigma_2^{248} \cong \mathbb{C}P^{248} \cong \mathrm{Gr}(1, \mathbb{C}^{249})$ (Kendall, 1984).

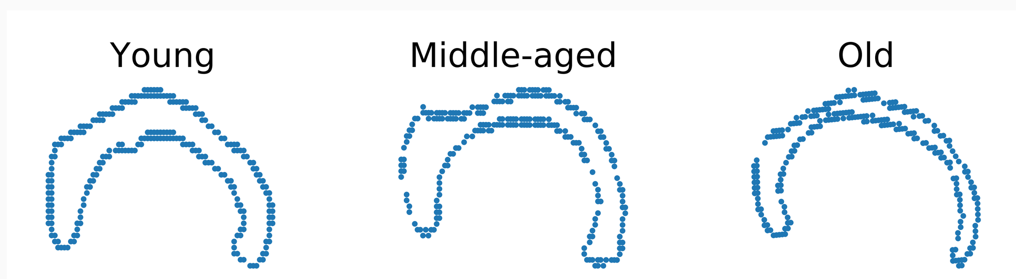

Figure 4: Example Corpus Callosi shapes from three distinct age groups, each depicted using the boundary point sets.

OASIS Datasets Experiments – Results

- For the unsupervised case, we seek a submanifold with (complex) dimension m, i.e. for NG, we project to $\mathrm{Gr}(1, \mathbb{C}^{m+1})$ and for PGA, we take first m principal components.
- For the supervised case, we fix $m = 10$ and apply the geodesic k-nearest neighbor (gKNN) algorithm after reducing the dimension.

	m				
	1	5	10	15	20
NG	26.38%	68.56%	84.18%	90.63%	94.04%
PGA	7.33%	43.74%	73.48%	76.63%	79.9%
PNSS	16.53%	51.68%	71.57%	82.49%	89.45%

Table 2: Percentage of explained variance by PGA and NG representations respectively. The PNSS stands for Principal nested shape space by Dryden et al. (2019).

	Accuracy	Explained Var.
gKNN	33.33%	N/A
gKNN + sPGA	38.89%	3.27%
gKNN + sNG	66.67%	98.7%
gKNN + PGA	30.56%	46.61%
gKNN + NG	30.56%	84.28%

Table 3: Classification accuracies and explained variances for sPGA and sNG.

References

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